$$d\theta_0(1)/dZ = 0 \quad (8c)$$

The first-order problem is

$$d^2\theta_1/dZ^2 - \lambda^2\theta_1 = -\theta_e^4 + \theta_0^4 \quad (9a)$$

subject to

$$\theta_1(0) = 0 \quad (9b)$$

$$d\theta_1(1)/dZ = -\nu[\theta_0^4(1) - \theta_e^4] \quad (9c)$$

The zero-order problem (8) is a nonradiating fin whose solution is

$$\theta_0(Z) = [(1 - \theta_e)/\cosh\lambda]\cosh\lambda(1 - Z) + \theta_e \quad (10)$$

The solution to the first-order problem (9) is

$$\theta_1(Z) = c_1e^{\lambda Z} + c_2e^{-\lambda Z} + \sum_{j=2}^{4} a_j \cosh j\lambda(1 - Z) + a_1Z\sinh\lambda(1 - Z) + c \quad (11)$$

where

$$a_1 = -(4\theta_e\alpha/\lambda)(3\alpha^2 + \theta_e^2),\ a_2 = (2\alpha^2/\lambda^2)(\tfrac{4}{3}\alpha^2 + 2\theta_e^2)$$

$$a_3 = \theta_e\alpha^3/\lambda^2,\ a_4 = \tfrac{2}{15}\alpha^4/\lambda^2,\ c = -(6\alpha^2/\lambda^2)(\alpha^2 + 2\theta_e^2)$$

$$c_1 = c_2e^{-2\lambda} + \psi_2,\ c_2 = (\psi_1 - \psi_2)/(1 + e^{-2\lambda})$$

$$\alpha = \frac{1}{2}\left(\frac{1 - \theta_e}{\cosh\lambda}\right)$$

$$\psi_1 = -c - \sum_{j=2}^{4} a_j \cosh j\lambda$$

$$\psi_2 = e^{-\lambda}[a_1 - (8\nu/\lambda)(2\alpha^4 + 4\theta_e\alpha^3 + 3\theta_e^2\alpha^2 + \theta_e^3\alpha)]$$

The complete solution, expressed in terms of Eqs. (7, 10, and 11), gives the temperature profile along the fin in terms of the radiation-conduction parameter ϵ, convection parameter (Biot modulus) λ, environment temperature θ_e, and fin geometry ν. The simple form of the solution permits temperature calculations without recourse to a computer and enables evaluation of the effects that problem parameters $(\epsilon,\lambda,\theta_e,\nu)$ have on the response of the fin.

To check the accuracy of the perturbation solution, the problem was also solved numerically. Figure 1 gives temperature profiles for various values of the perturbation parameter ϵ and Biot modulus λ^2, for $\theta_e = 0.1$ and $\nu = 0.0208$. As expected, the perturbation solution becomes increasingly more accurate as the perturbation parameter ϵ decreases, since for small ϵ, the nonlinear radiation effects are small relative to conduction. The accuracy of the solution is also observed to improve as the Biot modulus λ^2 is increased. For large λ, radiation effects become small relative to convection and the accuracy of the perturbation solution again improves.

References

[1] Bartas, J. G. and Sellers, W. H., "Radiation Fin Effectiveness," *Transactions of the ASME, Journal of Heat Transfer, Series C*, Vol. 82, 1960, pp. 73–75.

[2] Russell, L. D. and Chapman, A. J., "Analytical Solution for Transient Flow of Energy in a One-Dimensional Radiating Fin," *AIAA Journal*, Vol. 6, No. 1, Jan. 1968, pp. 90–93.

[3] Zerkle, R. D. and Sunderland, J. E., "The Transient Temperature Distribution in a Slab Subject to Thermal Radiation," *Transactions of the ASME, Journal of Heat Transfer, Series C*, Vol. 87, 1965, pp. 117–130.

[4] Lieblein, S., "Analysis of Temperature Distribution and Radiate Heat Transfer along a Rectangular Fin of Constant Thickness," TN D-196, 1959, NASA.

[5] Tien, C. L. and Abu-Romia, M. M., "Perturbation Solution in the Differential Analysis of Radiation Interactions with Conduction and Convection," *AIAA Journal*, Vol. 4, No. 4, April 1966, pp. 732–733.

[6] Dicker, D. and Asnani, M., "A Perturbation Solution for the Non-Linear Radiation Heat Transfer Problem," *Proceedings of the International Heat Transfer Conference*, Vol. 5, Chicago, Ill., 1966, pp. 164–173.

[7] Mueller, H. F. and Malmuth, N. D., "Temperature Distribution in Radiating Heat Shields by the Method of Singular Perturbation," *International Journal of Heat and Mass Transfer*, Vol. 8, 1965, pp. 915–920.

Piston-Retardation Insert in a Free-Piston Compressor

STELLAN KNÖÖS*

Aeronautical Research Institute of Sweden (FFA), Stockholm, Sweden

Introduction

FREE-PISTON compressors are commonly used for generating high temperatures and pressures in gases.[1,2] For shock-tube applications, the shock-tube driver gas is heated in the free-piston compressor and released into the driven shock tube by the opening of a high-pressure diaphragm. Due to experienced variations in diaphragm rupture pressures and difficulties in estimating peak pressures for given gas-loading conditions, the used diaphragms are allowed to break at pressures typically 20% lower than corresponding closed-end peak pressures. In this case, the piston possesses residual kinetic energy which has to be absorbed, normally as deformation work in some part of the mechanical structure. With incorrect initial gas loading or with a diaphragm weaker than normal, the residual energy could be larger than anticipated and mechanical destruction of the high-pressure section and the piston could result. In experiments with a free-piston compressor in a so-called bypass piston tube,[3] a sonic orifice has been inserted near the high-pressure diaphragm and successfully used in preventing such a high-speed impact. The insert, referred to as a piston-retardation insert, also favorably increases the entropy of the gas during the compression. Properly operating, e.g., in the case of an early rupture of the diaphragm, the piston will bounce in front of the insert or smoothly dock to the insert, with the gas between

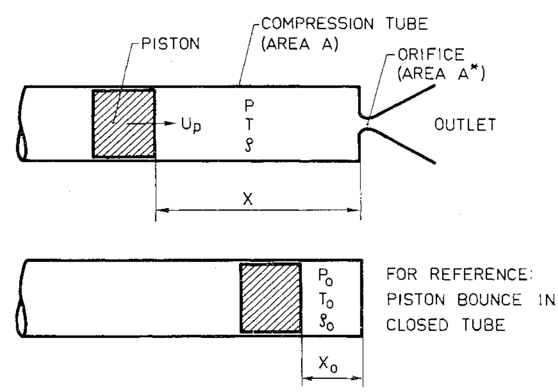
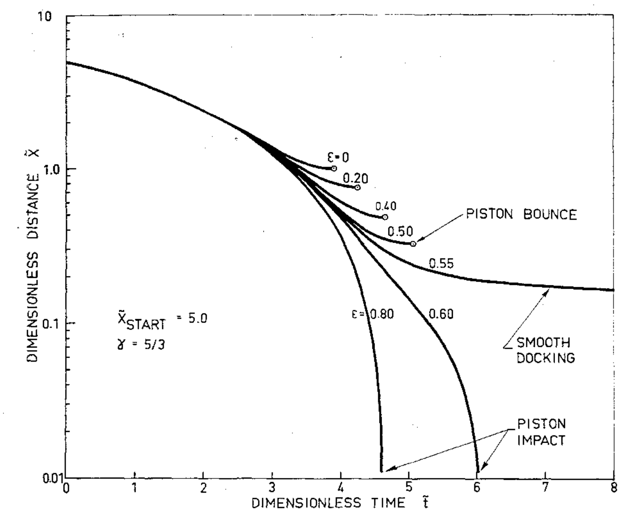


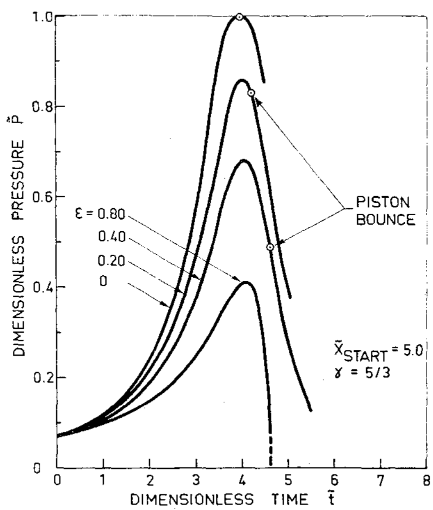
Fig. 1 Schematic view of the high-pressure section of a free-piston compressor with a sonic-orifice outlet. For reference is shown a piston bounce in a closed compression tube.

Received June 26, 1969; revision received August 21, 1969. Financial support from the Swedish Board for Technical Development (Styrelsen för Teknisk Utveckling), Contract 68-37/U36 is gratefully acknowledged.

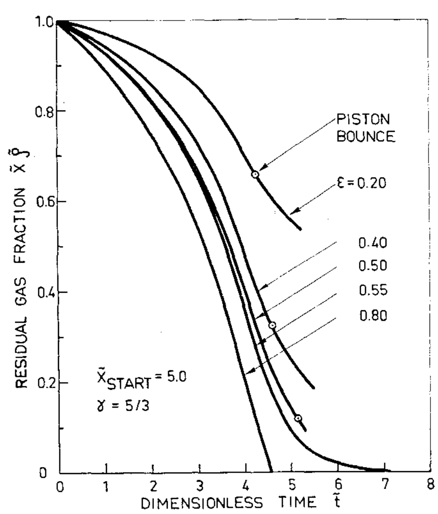
* Visiting scientist; presently with Northrop Corporate Laboratories, Hawthorne, Calif.



a) Dimensionless piston position $\tilde{x}$ as a function of time for different values of the parameter ϵ. Smooth docking is achieved for $\epsilon \sim 0.55$



b) Dimensionless gas pressure $\tilde{p}$ as a function of time



c) Residual fraction of gas $\tilde{x}\tilde{p}$, trapped between the piston and the orifice, as a function of time

Fig. 2 Results of numerical calculations for free-piston compression with a sonic-orifice flow outlet. Perfect gas with $\gamma = 5/3$. Orifice opening at $\tilde{x}_s = 5$.

the piston and the insert flowing through the orifice of the insert at sonic velocity.

Analysis

The size of the orifice should be properly matched to relevant bounce parameters and can be determined from an analysis of a free-piston compression with a sonic flow outlet, as follows. Denote the orifice area A^* and the cross section of the compression tube A, as shown in Fig. 1. Let M be the piston mass, u_p the piston velocity, $u_p = -dx/dt$, and x the distance to the insert. Reference conditions at an ideal isentropic closed-tube bounce have subscript 0. For simplicity, we limit the study to a stage of the compression when the pressure accelerating the piston can be neglected when compared to the pressure p of the gas in front of the piston. This is always true during the final part of the compression stroke. Assume a perfect gas, homogeneous thermodynamic conditions in the gas, and isentropic processes; the momentum equation for the motion of the piston and the equation for continuity of mass of the gas can then be written

$$d^2\tilde{x}/d\tilde{t}^{\,2} = \tilde{p}^{\gamma} \tag{1}$$

and

$$d\tilde{p}\tilde{x}/d\tilde{t} = -\epsilon\tilde{p}^{(\gamma+1)/2} \tag{2}$$

respectively. Here γ is the ratio of specific heats, $\tilde{x}$ the nondimensional piston distance to the orifice defined as $\tilde{x} = x/x_0$, $\tilde{p}$ the nondimensional gas density defined as $\tilde{p} = \rho/\rho_0$, and $\tilde{t}$ a nondimensional time defined as $\tilde{t} = t/\tau_0$, where τ_0 is a characteristic time for the bounce;

$$\tau_0 = (Mx_0/\rho_0 A)^{1/2} \tag{3}$$

The dimensionless parameter ϵ is defined as $\epsilon = \tau_0/\tau_1$, where τ_1 is a characteristic time

$$\tau_1 = \left(\frac{\gamma+1}{2}\right)^{(\gamma+1)/2(\gamma-1)} \frac{A}{A^*}\frac{x_0}{a_0} \tag{4}$$

for the sonic gas flow leaving a volume x_0A, relevant to a closed-tube bounce. The speed of sound at the reference bounce condition is denoted a_0.

Optimum performance for the shock-tube driver application can be achieved for a narrow range of ϵ values for which the piston reaches the position $\tilde{x} = 0$ with negligible residual velocity or it actually bounces at a small distance from the insert with $\tilde{x} << 1$.

For numerically practical reasons, the orifice was assumed to open at time $\tilde{t} = 0$ for a piston location $\tilde{x}_s = 5$, instead of being continuously open. The finite-difference calculations could then be started at this point, since for earlier piston motion, a relevant analytical solution was available from

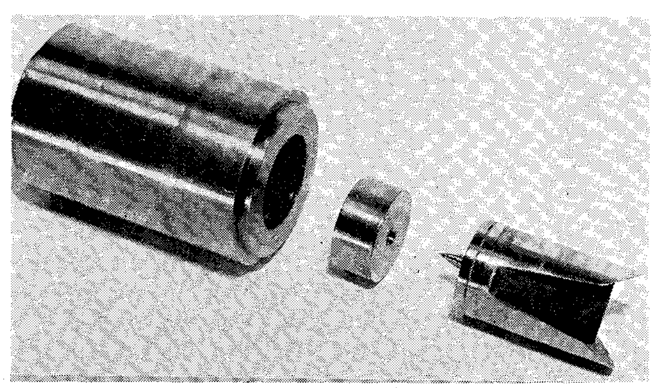


Fig. 3 Photograph of the disassembled high-pressure section of the compression tube, the piston retardation insert, and the free piston used in the bypass-piston shock-tube experiments.

Eq. (1). Hence, the piston velocity at $\tilde{x} = \tilde{x}_s$ was found to be

$$\left.\frac{d\tilde{x}}{dt}\right|_s = -\left[\frac{2}{\gamma - 1}\left(1 - \tilde{x}_s^{-(\gamma - 1)}\right)\right]^{1/2} \quad (5)$$

and corresponding gas density

$$\tilde{\rho}_s = 1/\tilde{x}_s \quad (6)$$

In Fig. 2a are shown numerical results for the dimensionless distance $\tilde{x}$ as a function of dimensionless time $\tilde{t}$ for discrete values of ϵ in the range $0 \leq \epsilon \leq 0.80$ for a gas with $\gamma = \frac{5}{3}$. For $\epsilon = 0$, we have the case of a closed orifice and the bounce takes place exactly at $\tilde{x} = 1$. For increasing values of ϵ, the bounce will occur closer to the insert and at later times. With $\epsilon \sim 0.55$ (for $\tilde{x}_s = 5$), smooth docking is obtained. Piston impact will happen for larger values of ϵ, as shown for $\theta = 0.60$ and $\epsilon = 0.80$. Corresponding pressure-time characteristics are drawn in Fig. 2b. The peak gas pressure in the bounce is found to take smaller values with increasing values of ϵ. Interestingly, the actual reversal of the piston velocity (the bounce) takes place after the pressure has reached its maximum value. In Fig. 2c, the nondimensional measure $\tilde{x}\tilde{\rho}$ of the amount of trapped gas is given as a function of time. Naturally, for $\epsilon = 0$, we have $\tilde{x}\tilde{\rho} = 1$. Values for $\tilde{x}\tilde{\rho}$ decrease continuously with time for all values $\epsilon > 0$. The case of smooth docking for $\epsilon \sim 0.55$ shows a negligible amount of residual gas for $\tilde{t} > 7$.

Experiment and Concluding Remarks

In experiments conducted with a bypass-piston shock tube,[4] the sonic-orifice insert was placed in the helium compression tube at a location near the turning point of the piston in a calculated bounce. In Fig. 3 is shown a photograph of the employed high-pressure section, the insert (65-mm o.d.) and a piston. Typical bounce parameters were for helium: $x_0 \sim 40$ mm, $p_0 \sim 600$ atm, and $a_0 \sim 5500$ m/s (temperature $T_0 \sim 9000$°K). The pistons used had masses in the range $0.4 < M < 2.0$ kg, and therefore $0.3 \leq \tau_0 \geq 0.6$ ms. The effective area ratio was $A/A^* \sim 35$, giving $\tau_1 \sim 0.5$ ms, and hence $0.6 \leq \epsilon \leq 1.2$. These values for ϵ were larger than the calculated critical $\epsilon \sim 0.55$ for smooth docking with $\tilde{x}_s = 5$. However, the heavy pistons (with an associated larger value for ϵ) were equipped with conical forebodies with a base diameter equal to the diameter of the orifice in the insert, as is visible in Fig. 3. By gradually plugging the orifice, these cones reduced the effective value of ϵ when the piston was in the immediate vicinity of the insert and thus diminished the possibility of impact.

The insert was frequently used under conditions which would normally have resulted in destructive piston impacts into the smaller-diameter high-pressure section. With the insert, no such impacts were observed, nor did the steel pistons swell, jam, or seize to the compression tube. The function of the insert was considered most satisfactory and vitally helpful in the successful attempts of achieving shock velocities above 12 km/s in the shock tube.[4]

As theoretically predicted and experimentally verified, the sonic-orifice insert in the free-piston shock-tube driver could completely avoid piston-impact hazards generally connected with a premature rupture of the outlet diaphragm or incorrect gas loading in a free-piston compressor. Use of the insert improves theoretical over-all performance due to an associated increase in entropy of the gas. For a properly designed orifice, a preferred value for the dimensionless parameter ϵ should be somewhat larger than or equal to the calculated value for smooth docking.

References

[1] Stalker, R. J., "The Free Piston Shock Tube," *Aeronautical Quarterly*, Nov. 1965, p. 351.

[2] Stalker, R. J., "A Study of the Free-Piston Shock Tunnel," *AIAA Journal*, Vol. 5, No. 12, Dec. 1967, pp. 2160–2165.

[3] Knöös, S., "Bypass Piston Tube, a New Device for Generating High Temperatures and Pressures in Gases," *AIAA Journal*, Vol. 6, No. 4, April 1968, pp. 632–641.

[4] Knöös, S., "Study of a Bypass Piston Shock Tube," *Proceedings of the Seventh International Shock Tube Symposium*, June 1969, Toronto, Canada.

Maximum-Minimum Sufficiency and Lagrange Multipliers

Thomas L. Vincent* and Eugene M. Cliff†
University of Arizona, Tucson, Ariz.

I. Introduction

LAGRANGE multipliers are commonly employed when dealing with problems of maxima and minima subject to equality constraints. Their use is usually justified in accordance with some multiplier rule‡ generally phrased in such a way that it provides a convenient constructive formalism for solving problems.

The proof of a multiplier rule found in texts, in particular advanced engineering texts such as Hildebrand (Ref. 1, p. 354), Solkolnikoff, and Redheffer (Ref. 2, p. 256), etc., is often very limited and quite intuitive. Often the general case is not proven and almost universally a discussion of sufficiency conditions for constrained extrema in terms of Lagrange multipliers is omitted.

Because optimizing conditions can be stated very simply and concisely in terms of Lagrange multipliers they remain in popular use. However, perhaps partly because of this extensive use, and partly because of the intuitive notions frequently offered, certain properties have at times been ascribed to the Lagrange multipliers which they do not possess. If we wish to extremize a function $f(x_i)$, $i = 1, \ldots, n$, subject to the constraints $g_j(x_i)$, $j = 1, \ldots, m < n$, a common misconception is the thought that this is identical to extremizing an augmented function $G(x_i,\lambda_j)$ formed by adjoining the constraints of f with the multipliers λ_j, i.e., $G = f + \lambda_j g_j$. For example, Edelbaum (Ref. 3, p. 11) in his discussion of theory of maxima and minima states, "The use of the augmented function allows a problem with subsidiary conditions to be replaced by a problem without subsidiary conditions. This new problem is amenable to all of the techniques used for solving problems without subsidiary conditions, including sufficiency conditions." Edelbaum is by no means the only author who alludes to this concept; but perhaps he states it most clearly.

Edelbaum's statement in essence represents a multiplier rule. To illustrate the results obtained by using this rule, we first need the following theorem for unconstrained extrema. For proof see Hestenes (Ref. 4, p. 18).

Theorem 1

The necessary and sufficient conditions for the function $f(x_i)$ of class C^2 on S to take on a local, proper, nonsingular minimum at the point x_i^p on the interior of S is that

$$\partial f/\partial x_i|_{x_i^p} = 0 \quad (1)$$

Received July 7, 1969; revision received August 25, 1969. This work was supported in part by NASA Grant No. NSG-580.

* Professor of Aerospace and Mechanical Engineering. Member AIAA.

† Assistant Professor of Aerospace and Mechanical Engineering. Member AIAA.

‡ A very loose usage is meant here. In other words, a multiplier rule is any rule which postulates optimizing conditions in terms of Lagrange multipliers. Associate AIAA.